

OPEN AND SOLVED PROBLEMS IN INFINITE COMBINATORICS

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ABSTRACT. We list some open problems, concerning the polarized partition relation. We solve a couple of them, by showing that for every singular cardinal μ there exists a forcing notion \mathbb{P} such that the strong polarized relation $(\mu^+)_\mu \rightarrow (\mu^+)_2^{1,1}$ holds in $\mathbf{V}^\mathbb{P}$.

Nous passons en revue certains problèmes non résolus concernant la relation de partition polarisée. Nous en résolvons deux en montrant que pour chaque cardinal singulier μ , il existe un forcing \mathbb{P} tel que $\mathbf{V}^\mathbb{P}$ satisfait la relation polarisée forte $(\mu^+)_\mu \rightarrow (\mu^+)_2^{1,1}$.

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0. INTRODUCTION

The central notion in this paper is the following:

Definition 0.1. The strong polarized relation.

The strong polarized relation $(\lambda^\lambda)_\kappa \rightarrow (\lambda^\lambda)_2^{1,1}$ means that for every function $c : \lambda \times \kappa \rightarrow 2$ there are $A \subseteq \lambda$ and $B \subseteq \kappa$ such that $|A| = \lambda$, $|B| = \kappa$ and $c \upharpoonright (A \times B)$ is constant.

The strong polarized relation $(\lambda^+) \rightarrow (\lambda^+)_2^{1,1}$ has been established many years ago, for the specific case of $\lambda = \aleph_0$. Until recent years, it was open whether any uncountable cardinal may satisfy this relation. Moreover, even weaker relations of this kind were unknown to exist on uncountable cardinals. See, for example, [14] which asks whether there exists any uncountable cardinal κ such that $(\kappa^+) \rightarrow (\kappa^+)_2^{1,1}$.

By previous papers we know to get the strong relation $(\lambda^+) \rightarrow (\lambda^+)_2^{1,1}$ for every supercompact cardinal λ (see [11]). For a singular cardinal μ , we introduced a positive consistency result under the assumption that μ is a limit of measurable cardinals (see [10]). An improvement of the method yielded the same for singulars which are limit of inaccessible cardinals, or even limit of strong limit cardinals. Of course, more properties are required for this consistency result, but one can force it. This appears in [11].

The process of replacing measurability by merely strong inaccessibility, excavated the fact that large cardinals are not really needed for these strong results. What we really need is captured in the following metamathematical idea, which serves in many other theorems:

*Whenever GCH yields a negative result
an appropriate pcf assumption
may give a positive result*

The appropriate pcf assumption is not always at hand. But in our case it is possible to force the required assumption for every singular cardinal. Suppose μ is a singular cardinal. We shall prove that the strong polarized relation $(\mu^+) \rightarrow (\mu^+)_2^{1,1}$ is consistent with ZFC (by ZFC we include the assumption that a supercompact cardinal exists in the ground model). This holds, in particular, for the baby case of $\mu = \aleph_\omega$.

Before seeking into the detailed proofs, let us try to explain what we really need for the strong polarized relation. Let μ be a singular cardinal, and let κ be the cofinality of μ . We may assume that μ is strong limit, and enough GCH holds below μ . This can be forced for every μ , and we shall need it in the proof. By a straightforward generalization of Theorem 1.1 we know that for getting the positive relation $(\mu^+) \rightarrow (\mu^+)_2^{1,1}$ we must increase 2^μ . In the case of countable cofinality we can do it with full GCH below μ (see [18], [19] and [20]), and in the case of uncountable cofinality we will maintain

enough GCH below μ to provide the combinatorial result on the one hand, and the ability to increase 2^μ on the other hand.

Given a coloring $c : \mu^+ \times \mu \rightarrow \theta$ we are trying to create a monochromatic product $A \times B$. We call A the big component (its cardinality being μ^+), and B is the small component respectively. As the size of B is the singular cardinal μ , it is quite natural to compound B from tiny pieces. So we fix a sequence of regular cardinals $\langle \mu_\varepsilon : \varepsilon < \kappa \rangle$ which tends to μ . Our pcf assumptions will focus mainly on this sequence.

At first stage we deal with the small component B , so let us fix an ordinal $\alpha < \mu^+$, and concentrate in the restriction $c \upharpoonright \{\alpha\} \times \mu$. Moreover, we shall work with every μ_ε separately, so we concentrate in $c \upharpoonright \{\alpha\} \times \mu_\varepsilon$. For every $\gamma < \mu_\varepsilon$ we have a value $c(\alpha, \gamma)$, which is a color below θ . Inasmuch as θ is small (and μ_ε is regular) we may pick a large set B_ε of size μ_ε so that $c(\alpha, \gamma)$ is the same for every $\gamma \in B_\varepsilon$. Without loss of generality, the color is 0.

The same process can be rendered for every $\varepsilon < \kappa$. Of course, the unified color for each B_ε may vary from one ε to another. But again, inasmuch as $\theta < \kappa = \text{cf}(\kappa)$ we shall get κ -many B_ε -s with the same color. So we assume, without loss of generality, that the color is 0 and it holds for each $\varepsilon < \kappa$. Set $B = \bigcup \{B_\varepsilon : \varepsilon < \kappa\}$, and the desired small component is at hand, with respect to our fixed $\alpha < \mu^+$.

We turn now to the big component, the set A . Fix for awhile some $\varepsilon < \kappa$. We can isolate some $B_\varepsilon \in [\mu_\varepsilon]^{\mu_\varepsilon}$ for each $\alpha < \mu^+$, but notice that B_ε depends on α , so it would be better to call it B_ε^α . Having different B_ε^α -s (for different α -s) is problematic, but μ^+ is much larger than the number of possible B_ε^α -s. Indeed, we have at most θ^{μ_ε} options, and by the strong limitude of μ we know that $\theta^{\mu_\varepsilon} < \mu < \mu^+$. It follows that there is a set A_ε of cardinality μ^+ and a fixed set B_ε of cardinality μ_ε so that $c \upharpoonright (A_\varepsilon \times B_\varepsilon) = 0$.

There is only one step which is required to accomplish the construction. We have to collect all the B_ε -s. But in order to do so, we need to ensure that A_ε is the same set for every $\varepsilon < \kappa$, or at least that the intersection of all the A_ε -s includes a set of size μ^+ . This is the last step, and it can never be accomplished in ZFC. In fact, if $2^\mu = \mu^+$ then the strong relation $\binom{\mu^+}{\mu} \rightarrow \binom{\mu^+}{\mu}_2^{1,1}$ fails. But let us try to understand what exactly prevents this last step.

In order to intersect κ -many sets, each of size μ^+ , and remain with a set of cardinality μ^+ , we need a κ^+ -complete ultrafilter on μ^+ . Such an ultrafilter exists if some compact cardinal is situated between κ and μ . But in this direction we also need that each A_ε is a set in the ultrafilter, and for this we need more and more completeness. If we give up the full relation for the small component, then we can get the following result:

Claim 0.2. *Polarized relation for a limit of compact cardinals.*

Assume $\mu > \text{cf}(\mu) = \kappa$, μ is a limit of compact cardinals.

Then $\binom{\mu^+}{\mu} \rightarrow \binom{\mu^+}{\beta}_2^{1,1}$, for every $\beta < \mu$. Moreover, for every $\beta < \mu$ there is a

uniform ultrafilter U_β on μ^+ so that the big component of the monochromatic product belongs to U_β .

Proof.

We shall use the notation of the discussion above this claim, so in particular a coloring c is given, and all the cardinals and sets defined above. Let β be an ordinal below μ . Let $\zeta < \kappa$ be the first ordinal such that $\beta < \mu_\zeta$, and denote θ^{μ_ζ} by τ . Choose a τ^+ -complete ultrafilter U_β on μ^+ . This is possible as μ is a limit of compact cardinals.

By the considerations above, one can choose B_ε -s and A_ε -s for every $\varepsilon < \zeta + 1$, such that $A_\varepsilon \in U_\beta$ for every $\varepsilon < \zeta + 1$. Set $A = \bigcap \{A_\varepsilon : \varepsilon < \zeta + 1\}$, and $B = \bigcup \{B_\varepsilon : \varepsilon < \zeta + 1\}$. It is easily verified that $A \times B$ is monochromatic for the coloring c , so the proof is complete. $\square_{0.2}$

As clearly seen, this argument will never be true if we replace "every $\beta < \mu$ " by μ itself. Indeed, for μ we need a μ^+ -complete ultrafilter on μ^+ , which is impossible. One might suspect that this obstacle cannot be overcome, and the strong polarized relation (with respect to the pair μ, μ^+) must fail.

But here, in the last step of creating the monochromatic product, we can invoke pcf theory. Under some pcf assumptions we can show that the sets A_ε are not arbitrary, and some properties are shared by every $\alpha < \mu^+$. This is exactly the missing part which is required to finish the construction of the full product.

The main idea is as follows. First we enumerate all the subsets of μ_ε , for every $\varepsilon < \kappa$. It helps (although not essential) if $2^{\mu_\varepsilon} = \mu_\varepsilon^+$. Consider now a fixed ordinal $\alpha < \mu^+$. The sets of the form B_ε appear in these enumerations, each B_ε at some place of the pertinent enumeration. This draws a function f_α in the product $\prod_{\varepsilon < \kappa} \mu_\varepsilon^+$. Repeating this process for every $\alpha < \mu^+$ we define μ^+ -many functions in the above product. Now pcf assumption enables us to find a single function f in this product which bounds every f_α . The assumption is that $\text{tcf}(\prod_{\varepsilon < \kappa} \mu_\varepsilon^+) \neq \mu^+$. Clearly, one has to increase 2^μ above μ^+ in order to get it, but nothing else is required.

How do we use this dominating function f ? Well, it depends on the extra assumptions that we add. In the simplest case we assume that each μ_ε is a measurable cardinal (that was the idea behind the main theorem in [10]). In this case we can arrange enough subsets of μ_ε in a \subseteq^* -decreasing chain, and just take the union of the B_ε -s which appear in the place $f(\varepsilon)$ for every $\varepsilon < \kappa$. In the more sophisticated theorems, culminating in the main theorem of the present paper, we iterate (finitely many times) such pcf arguments, and eliminate extra assumptions (like measurability) on the cardinals μ_ε -s.

We emphasize that the corollaries from the main theorem below are, in a sense, best possible. The number of colors (which is θ) must be below κ , the size of the monochromatic product cannot be larger of course, and the pcf

assumptions can be forced for every singular μ . We also indicate that the pair (μ, μ^+) is just a private case, and the theorem below is phrased for the general case of (μ, λ) when λ is any regular cardinal between μ and 2^μ . The strong polarized relation holds trivially for the pair (μ, λ) when λ exceeds 2^μ , and the behaviour of a singular λ is determined by that of $\text{cf}(\lambda)$.

The paper consists of two parts. In the first part we list some open problems concerning the polarized relation. In the second part (i.e., the last section) we provide a solution to some of them.

1. BACKGROUND

Most of the problems in the list below are based on the notion of strong polarized relations. We divide them into two groups. One section is devoted to \aleph_0 , and the second to uncountable cardinals. In this section we sketch known results, to be referred in the list of problems.

Polarized partition relations were introduced in [6], and investigated further in [5]. Our starting point is the following negative relation of Erdős, Hajnal and Rado:

Theorem 1.1. *If $2^{\aleph_0} = \aleph_1$ then $(\omega_1)_\omega \not\rightarrow (\omega_1)_2^{1,1}$.*

This negative result is not a theorem of ZFC. First, it is consistent that $(\omega_1)_\omega \rightarrow (\omega_1)_2^{1,1}$ (e.g., if $\text{MA} + 2^{\aleph_0} > \aleph_1$ holds, as proved in [15]). Second, by enlarging 2^{\aleph_0} we can get a positive relation of the form $(2^\omega)_\omega \rightarrow (2^\omega)_2^{1,1}$. It was noted, first, in [1] in the specific case of $2^{\aleph_0} = \aleph_2$ (in the model of iterating Sacks reals) and proved in a more general setting in [8] and [9]:

Theorem 1.2. *Suppose $\lambda > \aleph_1$ and $\lambda = \lambda^{\aleph_0}$ (hence $\text{cf}(\lambda) > \aleph_0$).*

Then *there is a forcing notion \mathbb{P} so that $\mathfrak{c} = \lambda$ and $(\lambda)_\omega \rightarrow (\lambda)_2^{1,1}$ in $\mathbf{V}^\mathbb{P}$.*

A similar positive result holds upon replacing \aleph_0 by a supercompact cardinal μ . This is also proved in [8]:

Theorem 1.3. *Suppose μ is a supercompact cardinal.*

Then *it is consistent that $(2^\mu)_\mu \rightarrow (2^\mu)_2^{1,1}$.*

A straightforward generalization of 1.1 asserts that $2^\kappa = \kappa^+$ implies $(\kappa^+)_\kappa \not\rightarrow (\kappa^+)_2^{1,1}$ for every infinite cardinal κ . As far as we know, a parallel positive result for κ and κ^+ under the assumption $2^\kappa > \kappa^+$ is known only under assumptions of large cardinals. The following concerns regular limit cardinals (its proof is included in [8]):

Theorem 1.4. *Suppose μ is a supercompact cardinal.*

Then $(\mu^+)_\mu \rightarrow (\mu^+)_\theta^{1,1}$ *for every $\theta < \mu$ is consistent with ZFC.*

A parallel result is proved in [10] for singular cardinals:

Theorem 1.5. *Suppose there is a supercompact cardinal in the ground model.*

Then *it is consistent that there is a singular cardinal μ , a limit of measurable cardinals, so that $(\mu^+)_\mu \rightarrow (\mu^+)_\theta^{1,1}$ for every $\theta < \text{cf}(\mu)$.*

The next theorem concerns the cardinals below the splitting number. The proof of the countable case appears in [8], and the proof of the general case is situated in [11]. Here is the assertion:

Theorem 1.6. *Suppose $\kappa < \lambda < \mathfrak{s}_\kappa$.*

The positive relation $(\lambda)_\kappa \rightarrow (\lambda)_2^{1,1}$ holds iff $\text{cf}(\lambda) \neq \kappa$.

The last theorem (from [22]) to be mentioned in this section is almost strong. It gives a balanced form of the polarized relation, but the size of the big component can be $\alpha < \mu^+$ for every $\alpha < \mu^+$, and not μ^+ itself:

Theorem 1.7. *Suppose μ is a limit of measurable cardinals.*

Then $(\mu^+)_\mu \rightarrow (\alpha)_\mu^{1,1}$ *for every $\alpha < \mu^+$.*

We state below some problems that remain unsolvable in light of these results. Many of them are connected with cardinal invariants. We refer to [2] for a comprehensive exposition of this subject. For a modern discussion on the polarized relation, see [12].

2. COUNTABLE PROBLEMS

Theorem 1.1 can be rephrased as follows: $\mathfrak{c} = \aleph_1$ implies $(\mathfrak{c})_\omega \not\rightarrow (\mathfrak{c})_2^{1,1}$. We may ask whether \mathfrak{c} is the correct cardinal invariant for this negative implication:

Problem 2.1. Suppose \mathfrak{x} is a nicely defined invariant which satisfies $\mathfrak{x} = \aleph_1 \Rightarrow (\mathfrak{x})_\omega \rightarrow (\mathfrak{x})_2^{1,1}$. Does it follow that $\mathfrak{x} = \mathfrak{c}$?

We are interested, in general, in the possibility that a cardinal invariant equals \aleph_1 yet a positive relation is consistent with respect to this invariant and \aleph_0 . If $\mathfrak{x} = \aleph_1 < \mathfrak{s}$ is consistent (e.g., $\mathfrak{x} = \mathfrak{a}$ or $\mathfrak{x} = \mathfrak{b}$) then $\mathfrak{x} = \aleph_1$ and $(\mathfrak{x})_\omega \rightarrow (\mathfrak{x})_2^{1,1}$ is consistent (see [8], Claim 2.4). So our problem applies to cardinal characteristics above the splitting number:

Problem 2.2. Is it consistent that $\mathfrak{s} = \aleph_1$ yet $(\mathfrak{s})_\omega \rightarrow (\mathfrak{s})_2^{1,1}$?

The (consistency of the) positive relation $(\mathfrak{s})_\omega \rightarrow (\mathfrak{s})_2^{1,1}$ is established in [11] in the case of $\mathfrak{s} = \aleph_2$. We may ask:

Problem 2.3. Is it consistent that $\mathfrak{s} > \aleph_2$ and $(\mathfrak{s})_\omega \rightarrow (\mathfrak{s})_2^{1,1}$?

Another variant of this problem is related to the ultrafilter number. The positive result of [11] comes from a model of Blass and Shelah (see [3]) in which $\mathfrak{u} < \mathfrak{s} = \mathfrak{c}$. Decreasing \mathfrak{u} (or even \mathfrak{r}) is one way to get upward positive relations. But one may wonder if this is essential:

Problem 2.4. Is it consistent that $\mathfrak{s} = \mathfrak{u}$ and still $(\mathfrak{s})_\omega \rightarrow (\mathfrak{s})_2^{1,1}$?

Or from the opposite point of view:

Problem 2.5. Is it consistent that $\mathfrak{s} < \mathfrak{c}$ and $(\mathfrak{s})_\omega \rightarrow (\mathfrak{s})_2^{1,1}$?

Notice that a positive answer to Problem 2.2 entails a positive answer to the present problem. As mentioned above, Problem 2.2 applies to every cardinal invariant above the splitting number. Our knowledge about the dominating number, in this respect, is slightly better. In particular, for every $\kappa = \text{cf}(\kappa) > \aleph_1$ we can prove the consistency of $\mathfrak{d} = \kappa \wedge (\mathfrak{d})_\omega \rightarrow (\mathfrak{d})_2^{1,1}$. The proof is spelled out in [11], and the model comes from [4]. This gives rise to the following problems:

Problem 2.6. Is it consistent that $\mathfrak{d} = \aleph_1$ and $(\mathfrak{d})_\omega \rightarrow (\mathfrak{d})_2^{1,1}$?

This problem is parallel to 2.2. It is an open problem whether \mathfrak{s} can be a singular cardinal. But it is known that \mathfrak{d} might be singular. In fact, if $\aleph_0 < \kappa = \text{cf}(\kappa) \leq \text{cf}(\lambda) \leq \lambda$ then $\mathfrak{b} = \kappa \wedge \mathfrak{d} = \lambda$ is consistent. The model of Blass and Shelah from [4] (in which $\mathfrak{u} < \mathfrak{d}$) requires the regularity of \mathfrak{d} . Hence the following may be phrased:

Problem 2.7. Is it consistent that \mathfrak{d} is a singular cardinal and $(\mathfrak{d})_\omega \rightarrow (\mathfrak{d})_2^{1,1}$?

More generally, we may try to eliminate the use of $\mathfrak{u} < \mathfrak{d}$ in order to get a positive relation. So the following (which is parallel to 2.4) makes sense:

Problem 2.8. Is it consistent that $\mathfrak{d} \leq \mathfrak{u}$ and $(\mathfrak{d}) \rightarrow (\mathfrak{d})_2^{1,1}$?

And back again, we can try to look at this problem from the opposite direction of $\mathfrak{d} < \mathfrak{c}$. Here is the parallel to Problem 2.5:

Problem 2.9. Assume $(\mathfrak{d}) \rightarrow (\mathfrak{d})_2^{1,1}$, and one adds λ -many random reals (for some $\lambda > \mathfrak{d}$). Is the positive relation $(\mathfrak{d}) \rightarrow (\mathfrak{d})_2^{1,1}$ still holds?

The last two problems in this vein are about the independence number:

Problem 2.10. Is it consistent that $\mathfrak{i} = \aleph_1$ and $(\mathfrak{i}) \rightarrow (\mathfrak{i})_2^{1,1}$?

And with respect to the ultrafilter number:

Problem 2.11. Is it consistent that $\mathfrak{i} \leq \mathfrak{u}$ and $(\mathfrak{i}) \rightarrow (\mathfrak{i})_2^{1,1}$?

We indicate that the forcing construction from [21], in which the consistency of $\mathfrak{i} < \mathfrak{u}$ is established, might be helpful for some of the above problems.

The following problem is related to real-valued measurable cardinals. In the original construction of Solovay (see [24]) it seems that such a cardinal carries a negative strong relation. But in [7] appears a different way to introduce a real valued measurable cardinal in the universe. We ask:

Problem 2.12. Let κ be a real valued measurable cardinal. Is it possible that $(\kappa) \rightarrow (\kappa)_2^{1,1}$?

Another direction of research concerns the spectrum of positive relations. Adding λ -many Cohen reals entails $(\kappa) \rightarrow (\kappa)_2^{1,1}$ for every $\kappa \in [\aleph_1, \aleph]$ (an explicit proof appears in [9], Remark 2.4). The opposite situation is consistent as well. In [11] it is proved that $(\kappa) \rightarrow (\kappa)_2^{1,1}$ for every $\kappa \in [\aleph_1, \aleph]$ is consistent in a model of $\mathfrak{c} = \aleph_2$. This invites the following:

Problem 2.13. Is it consistent that $\mathfrak{c} > \aleph_2$ and $(\kappa) \rightarrow (\kappa)_2^{1,1}$ for every $\kappa \in [\aleph_1, \aleph]$?

This can be phrased in a slight stronger version. As mentioned above, if $\kappa < \mathfrak{s}$ and $\text{cf}(\kappa) > \aleph_0$ then $(\kappa) \rightarrow (\kappa)_2^{1,1}$. It follows that if $\mathfrak{s} = \mathfrak{c} < \aleph_\omega$ then $(\kappa) \rightarrow (\kappa)_2^{1,1}$ for every $\kappa \in [\aleph_1, \aleph]$. We may ask:

Problem 2.14. Is it consistent that $\mathfrak{s} = \mathfrak{c} > \aleph_2$ and $(\kappa) \rightarrow (\kappa)_2^{1,1}$ for every $\kappa \in [\aleph_1, \aleph]$ (i.e., including $\kappa = \mathfrak{s}$ itself)?

Of course, $\mathfrak{c} < \aleph_\omega$ in such models, as $(\aleph_\omega) \rightarrow (\aleph_\omega)_2^{1,1}$. We say that \aleph_ω is a breaking point in the positive spectrum of strong polarized relations. The following is a natural question:

Problem 2.15. Is it consistent that \aleph_ω is the only breaking point? More generally, can \mathfrak{c} be arbitrarily large and the breaking points of the positive spectrum are just the cardinals of countable cofinality?

By iterating Mathias forcing notions (relativised to some ultrafilter) we can get $\mathfrak{c} = \aleph_n$ and a positive spectrum which includes only one breaking point (see [8]). One may ask if the positive spectrum can behave in a more complicated way:

Problem 2.16. Is it consistent that the polarized spectrum includes two breaking points below \aleph_ω ?

Observe that if $\theta_0 < \theta_1 < \aleph_\omega$ are such points then $\mathfrak{s} \leq \theta_0$ and $\mathfrak{u} \geq \theta_1$. By and large, the splitting number is responsible (to a large extent) for the nature of this spectrum. We may wonder about invariants that might be located below \mathfrak{s} and their relationship with \mathfrak{s} . The following is a typical question:

Problem 2.17. Is it consistent that $\mathfrak{a} = \mathfrak{s}$ and $\binom{\mathfrak{a}}{\omega} \rightarrow \binom{\mathfrak{a}}{\omega}_2^{1,1}$?

The same can be phrased with respect to invariants that are always below the splitting number. It is proved in [13] that $\binom{\mathfrak{p}}{\omega} \rightarrow \binom{\mathfrak{p}}{\omega}_2^{\alpha,1}$ for every $\alpha < \mathfrak{p}$. Due to 1.6, if $\mathfrak{p} < \mathfrak{s}$ then $\binom{\mathfrak{p}}{\omega} \rightarrow \binom{\mathfrak{p}}{\omega}_2^{1,1}$. The following is natural:

Problem 2.18. Can one prove that $\binom{\mathfrak{p}}{\omega} \rightarrow \binom{\mathfrak{p}}{\omega}_2^{1,1}$ iff $\mathfrak{p} < \mathfrak{s}$?

A starting point might be the case of $\mathfrak{p} = \mathfrak{s} = \mathfrak{c}$. In the second stage, we may try to treat $\mathfrak{p} = \mathfrak{s} < \mathfrak{c}$ by collapsing \mathfrak{c} using Lévy collapse from [17]. This invites a general problem:

Problem 2.19. Suppose $\aleph_1 < \kappa < \lambda = \mathfrak{c}$ and $\binom{\kappa}{\omega} \rightarrow \binom{\kappa}{\omega}_2^{1,1}$. Let $\mathbb{P} = \text{Levy}(\kappa, \lambda)$ be the Levy collapse. Is it possible that $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}_2^{1,1}$ in $\mathbf{V}^{\mathbb{P}}$?

Notice that if $\kappa = \aleph_1$ then the negative relation in $\mathbf{V}^{\mathbb{P}}$ follows. But what about the slightly weaker relation:

Problem 2.20. Suppose $\kappa < \lambda = \mathfrak{c}$ and $\binom{\kappa}{\omega} \rightarrow \binom{\kappa}{\omega}_2^{1,1}$. Let $\mathbb{P} = \text{Levy}(\kappa, \lambda)$ be the Levy collapse. Is it possible that for some ordinal $\alpha < \kappa$ we have $\binom{\kappa}{\omega} \nrightarrow \binom{\kappa}{\omega}_2^{\alpha,1}$ in $\mathbf{V}^{\mathbb{P}}$?

We conclude this section with a problem of a different kind. It is proved in [9] that the positive relation $\binom{\omega_1}{\omega} \rightarrow \binom{\omega_1}{\omega}_2^{1,1}$ is consistent with the weak diamond. We do not know what happens under the club principle:

Problem 2.21. Is it consistent that \clubsuit_{\aleph_1} holds and $\binom{\omega_1}{\omega} \rightarrow \binom{\omega_1}{\omega}_2^{1,1}$?

Observe that a positive answer requires \clubsuit_{\aleph_1} with the negation of the continuum hypothesis. The consistency of this setting is proved in [23], but the forcing construction is a bit sophisticated. It points out that a positive answer would not be easy.

3. UNCOUNTABLE CARDINALS

Recall that $\binom{2^\omega}{\omega} \rightarrow \binom{2^\omega}{\omega}_2^{1,1}$ has been generalized in [8] to every supercompact cardinal (see 1.3 above). The proof is based on the generalized Mathias forcing. Our ability to generalize Mathias forcing (relativised to some ultrafilter) requires the measurability of μ . For iterating without destroying the measurability, we employ a Laver-indestructible supercompact μ (see [16]). It feels that supercompactness is overwhelming, yet some large cardinals assumption is needed. So we phrase the following problems:

Problem 3.1. Let κ be a successor cardinal. Is the negative relation $\binom{2^\kappa}{\kappa} \not\rightarrow \binom{2^\kappa}{\kappa}_2^{1,1}$ a theorem of ZFC?

The same can be phrased for ‘small’ large cardinals, in particular the ones below weakly compact:

Problem 3.2. Suppose κ is inaccessible but not weakly compact. Is the negative relation $\binom{2^\kappa}{\kappa} \not\rightarrow \binom{2^\kappa}{\kappa}_2^{1,1}$ a theorem of ZFC?

In contrary, one may try to reduce the positive consistency result to a weaker assumption than supercompactness:

Problem 3.3. Let κ be a measurable (or strongly compact) cardinal. Is it consistent that $\binom{2^\kappa}{\kappa} \rightarrow \binom{2^\kappa}{\kappa}_2^{1,1}$?

Similar problems arise while replacing the pair $(\kappa, 2^\kappa)$ by the pair (κ, κ^+) . Generally speaking, a positive strong relation of the form $\binom{\lambda}{\kappa} \rightarrow \binom{\lambda}{\kappa}_2^{1,1}$ becomes harder to achieve when λ is closed to κ . For instance, if $\lambda = \kappa$ then a negative result follows for every infinite κ . Consequently, the possibility of $\binom{\kappa^+}{\kappa} \rightarrow \binom{\kappa^+}{\kappa}_2^{1,1}$ is an interesting problem.

By [15], Martin’s Axiom implies $\binom{\aleph_1}{\aleph_0} \rightarrow \binom{\aleph_1}{\aleph_0}_2^{1,1}$. More generally, if $\mathfrak{s} > \aleph_1$ then $\binom{\aleph_1}{\aleph_0} \rightarrow \binom{\aleph_1}{\aleph_0}_2^{1,1}$ (as appears in 1.6). It holds also for a supercompact cardinal κ , by forcing $\mathfrak{s}_\kappa > \kappa^+$. The consistency of $\mathfrak{s}_\kappa > \kappa$ requires κ to be a weakly compact cardinal (see [25]), so the following is natural:

Problem 3.4. Let κ be a weakly compact cardinal. Is it consistent that $\binom{\kappa^+}{\kappa} \rightarrow \binom{\kappa^+}{\kappa}_2^{1,1}$?

And back again, to the opposite direction:

Problem 3.5. Suppose $\kappa = \text{cf}(\kappa)$ is not a weakly compact cardinal. Is it provable in ZFC that $\binom{\kappa^+}{\kappa} \not\rightarrow \binom{\kappa^+}{\kappa}_2^{1,1}$?

The former problems deal with regular cardinals. In the case of a singular cardinal λ we can get a positive result according to 1.5. The proof requires that λ is a limit of measurable cardinals. One may try to eliminate this assumption:

Problem 3.6. Let λ be any strong limit singular cardinal. Is it consistent that $(\lambda^+) \rightarrow (\lambda^+)_2^{1,1}$? What about $(2^\lambda) \rightarrow (2^\lambda)_2^{1,1}$?

Of course, a violation of the (local) continuum hypothesis on λ must be employed. If $\lambda > \text{cf}(\lambda) = \aleph_0$ this is possible, even if the GCH holds below λ . We can ask:

Problem 3.7. Suppose $\lambda > \text{cf}(\lambda) = \aleph_0$, and $\theta < \lambda \Rightarrow 2^\theta = \theta^+$. Is it consistent that $(\lambda^+) \rightarrow (\lambda^+)_2^{1,1}$ or $(2^\lambda) \rightarrow (2^\lambda)_2^{1,1}$?

Weaker results may hold without violating the GCH. Recall that if λ is strong limit singular with $2^\lambda > \lambda^+$ then $(\lambda^+) \rightarrow (\lambda^{+1})_\theta^{1,1}$ for every $\theta < \text{cf}(\lambda)$ (this is proved in [22]). We can ask for the following strengthening:

Problem 3.8. Suppose $\lambda > \text{cf}(\lambda)$, and $2^\lambda > \lambda^+$. Is it provable in ZFC that $(\lambda^+) \rightarrow (\lambda^\alpha)_2^{1,1}$ for every $\alpha < \lambda^+$?

Let us return to the case of a singular cardinal λ being a limit of measurable cardinals. It is proved in [22] that $(\lambda^+) \rightarrow (\lambda^\alpha)_2^{1,1}$ for every $\alpha < \lambda^+$, in ZFC. As mentioned in the introduction, $(\lambda^+) \rightarrow (\lambda^+)_2^{1,1}$ fails to become a theorem of ZFC. But the intermediate situation remains open:

Problem 3.9. Suppose $\lambda > \text{cf}(\lambda)$ is a limit of measurable cardinals. Is it provable in ZFC that $(\lambda^+) \rightarrow (\lambda^+ \lambda^\alpha)_2^{1,1}$ for every $\alpha < \lambda^+$?

Even if this relation is not provable in ZFC, we may wonder if it is consistent with the local continuum hypothesis:

Problem 3.10. Suppose $\lambda > \text{cf}(\lambda)$ is a limit of measurable cardinals. Is it consistent that $2^\lambda = \lambda^+$ and yet $(\lambda^+) \rightarrow (\lambda^+ \lambda^\alpha)_2^{1,1}$ for every $\alpha < \lambda^+$?

On the other hand, one can get the impression that if λ is not a strong limit cardinal then a positive consistency result is denied:

Problem 3.11. Is it provable in ZFC that $(\lambda^+) \rightarrow (\lambda^+)_2^{1,1}$ whenever $\lambda > \text{cf}(\lambda)$ and λ is not strong limit?

We conclude with a spectral problem:

Problem 3.12. Let λ be a strong limit singular cardinal which is not a limit of measurable cardinals. Is it consistent that $\mu \in (\lambda, 2^\lambda] \Rightarrow (\mu) \rightarrow (\mu)_2^{1,1}$?

Under some extra assumptions on λ we can give a positive answer (see [11]). But the general case of a strong limit singular remains open.

4. DOWN TO \aleph_ω

The theorem below is phrased in the general context of the pair (μ, λ) . The specific cases of $\lambda = \mu^+$ and $\lambda = 2^\mu$ are more interesting (see Problems 3.6 and 3.7). We phrase the conclusions of this theorem, with respect to these cases.

Theorem 4.1. *The main theorem.*

Assume that:

- (a) $\mu > \text{cf}(\mu) = \kappa$, and μ is a strong limit cardinal
- (b) $\theta < \kappa$ (θ is the number of colors)
- (c) $\mu < \lambda = \text{cf}(\lambda) \leq 2^\mu$
- (d) $\langle \mu_\varepsilon : \varepsilon < \kappa \rangle$ is an increasing sequence of regular cardinals
- (e) $\mu = \bigcup \{\mu_\varepsilon : \varepsilon < \kappa\}$
- (f) $\mu_\varepsilon = \sum_{\alpha < \mu_\varepsilon} 2^{|\alpha|}$, for every $\varepsilon < \kappa$
- (g) $2^{\mu_\varepsilon} = \mu_\varepsilon^+$ and $2^{\mu_\varepsilon^+} = \mu_\varepsilon^{++}$ for every $\varepsilon < \kappa$
- (h) $\Upsilon_\ell = \text{tcf}(\prod_{\varepsilon < \kappa} \mu_\varepsilon^{+\ell})$ for $\ell \in \{0, 1, 2\}$
- (i) $\lambda \notin \{\Upsilon_0, \Upsilon_1, \Upsilon_2\}$

Then the strong polarized relation $(\lambda) \rightarrow (\mu)_2^{1,1}$ holds.

Proof.

Let $c : \lambda \times \mu \rightarrow \theta$ be any coloring. For every $\alpha < \lambda, \varepsilon < \kappa$ and $\iota < \theta$ we define the following set:

$$A_{\alpha, \varepsilon, \iota} = \{\gamma < \mu_\varepsilon^+ : c(\alpha, \gamma) = \iota\}$$

We enumerate the members of $\mathcal{P}(\mu_\varepsilon^+)$ as $\langle B_{\varepsilon, i} : i < \mu_\varepsilon^{++} \rangle$. For every $\alpha < \lambda$ we define a function $f_{\alpha, \iota} \in \prod_{\varepsilon < \kappa} \mu_\varepsilon^{++}$ by letting $f_{\alpha, \iota}(\varepsilon)$ be $\min\{i < \mu_\varepsilon^{++} : A_{\alpha, \varepsilon, \iota} \equiv B_{\varepsilon, i}\}$. Now we define, for every $\alpha < \lambda$ the function $f_\alpha \in \prod_{\varepsilon < \kappa} \mu_\varepsilon^{++}$ as follows:

$$f_\alpha(\varepsilon) = \sup\{f_{\alpha, \iota}(\varepsilon) : \iota < \theta\}$$

Observe that $f_\alpha \in \prod_{\varepsilon < \kappa} \mu_\varepsilon^{++}$ as θ is small and each μ_ε^{++} is regular. By assumption (i), $\lambda \neq \Upsilon_2$, hence one can choose a function $f \in \prod_{\varepsilon < \kappa} \mu_\varepsilon^{++}$ which bounds each f_α . It means that $\alpha < \lambda \Rightarrow f_\alpha <_{J_\kappa^{\text{bd}}} f$ for every $\alpha < \lambda$.

We may assume that $f(\varepsilon) \geq \mu_\varepsilon^+$ for every $\varepsilon < \kappa$. Without loss of generality, $f_\alpha(\varepsilon) < f(\varepsilon)$ for every $\alpha < \lambda$ and every $\varepsilon < \kappa$. We focus on $\{B_{\varepsilon, i} : i < f(\varepsilon)\}$, which is a subcollection of $\mathcal{P}(\mu_\varepsilon^+)$, for every $\varepsilon < \kappa$. By re-enumerating its members as $\{B_{\varepsilon, i}^1 : i < \mu_\varepsilon^+\}$ we can define (for every $\alpha < \lambda$ and $\iota < \theta$) a function $g_{\alpha, \iota} \in \prod_{\varepsilon < \kappa} \mu_\varepsilon^+$ by $g_{\alpha, \iota}(\varepsilon) = \min\{i < \mu_\varepsilon^+ : A_{\alpha, \varepsilon, \iota} \equiv B_{\varepsilon, i}^1\}$. As before, for every $\alpha < \lambda$ we define:

$$g_\alpha(\varepsilon) = \sup\{g_{\alpha,\iota}(\varepsilon) : \iota < \theta\}$$

Notice that $g_\alpha \in \prod_{\varepsilon < \kappa} \mu_\varepsilon^+$ as well. We employ, again, assumption (i) (this time we use $\lambda \neq \Upsilon_1$) to choose a bounding function $g \in \prod_{\varepsilon < \kappa} \mu_\varepsilon^+$. In this round, we assume that $g(\varepsilon) \geq \mu_\varepsilon$ for every $\varepsilon < \kappa$, and without loss of generality, $g_\alpha(\varepsilon) < g(\varepsilon)$ for every $\alpha < \lambda$ and every $\varepsilon < \kappa$.

For the last round, we re-enumerate the collection $\{B_{\varepsilon,i}^1 : i < g(\varepsilon)\}$ as $\{B_{\varepsilon,i}^2 : i < \mu_\varepsilon\}$. We define $h_{\alpha,\iota} \in \prod_{\varepsilon < \kappa} \mu_\varepsilon$ by $h_{\alpha,\iota}(\varepsilon) = \min\{i < \mu_\varepsilon : A_{\alpha,\varepsilon,\iota} \equiv B_{\varepsilon,i}^2\}$. And back again, for every $\alpha < \lambda$ we define:

$$h_\alpha(\varepsilon) = \sup\{h_{\alpha,\iota}(\varepsilon) : \iota < \theta\}$$

At this last stage we have $h_\alpha \in \prod_{\varepsilon < \kappa} \mu_\varepsilon$. By the last part of (i) we know that $\lambda \neq \Upsilon_0$, so a bounding function $h \in \prod_{\varepsilon < \kappa} \mu_\varepsilon$ is at hand. Without loss of generality, it bounds each h_α for every $\varepsilon < \kappa$.

Now for each $\varepsilon < \kappa$ we define an equivalence relation E_ε on the ordinals of μ_ε^+ as follows:

$$\gamma_0 E_\varepsilon \gamma_1 \Leftrightarrow \gamma_0 \in B_{\varepsilon,j}^2 \equiv \gamma_1 \in B_{\varepsilon,j}^2, \forall j < h(\varepsilon)$$

The number of equivalence classes is at most $2^{|h(\varepsilon)|} \leq \mu_\varepsilon < \mu_\varepsilon^+$. Choose (for every $\varepsilon < \kappa$) an equivalence class X_ε of E_ε of size μ_ε^+ . Observe that for every $\alpha < \lambda$ there is a color $\iota_{\alpha,\varepsilon} < \theta$ so that:

$$\gamma \in X_\varepsilon \Rightarrow c(\alpha, \gamma) = \iota_{\alpha,\varepsilon}$$

For every $\alpha < \lambda$ we choose a set $u_\alpha \in [\kappa]^\kappa$ and a fixed color $i_\alpha < \theta$ such that $\varepsilon \in u_\alpha \Rightarrow \iota_{\alpha,\varepsilon} = i_\alpha$. Since $2^\kappa < \lambda$ there is a set $A \in [\lambda]^\lambda$, a set $u \in [\kappa]^\kappa$ and a single color $\iota < \theta$ such that:

$$\alpha \in A \wedge \varepsilon \in u \Rightarrow \iota_{\alpha,\varepsilon} = \iota$$

Set $B = \bigcup\{X_\varepsilon : \varepsilon \in u\}$, so $|B| = \mu$. We claim that the product $A \times B$ is monochromatic for c . Indeed, if $\alpha \in A$ and $\beta \in B$ then there is an ordinal $\varepsilon \in u$ such that $\beta \in X_\varepsilon$, hence $c(\alpha, \beta) = \iota_{\alpha,\varepsilon} = i_\alpha = \iota$, and the proof is complete.

□_{4.1}

The main theorem is phrased in a general fashion. We may derive some specific conclusions from it. These conclusions are based on known forcing constructions for singular cardinals, mainly from [18],[19] and [20]. We quote the following:

Theorem 4.2. *The singular cardinals problem.*

Suppose μ is a supercompact cardinal, GCH holds above μ and $\delta < \mu_0 < \mu$. One can find a forcing notion \mathbb{P} which collapses cardinals only in the interval

$(\mu_0, \mu]$, making μ a singular cardinal of cofinality $\kappa = \text{cf}(\delta)$ and $\mu = \mu_0^{+\delta}$, such that in $\mathbf{V}^{\mathbb{P}}$ the following hold:

- (a) $\langle \mu_\varepsilon : \varepsilon < \kappa \rangle$ is an increasing sequence of regular cardinals
- (b) $\mu = \bigcup \{ \mu_\varepsilon : \varepsilon < \kappa \}$
- (c) $\mu_\varepsilon = \sum_{\alpha < \mu_\varepsilon} 2^{|\alpha|}$, for every $\varepsilon < \kappa$
- (d) $2^{\mu_\varepsilon} = \mu_\varepsilon^+$ and $2^{\mu_\varepsilon^+} = \mu_\varepsilon^{++}$ for every $\varepsilon < \kappa$
- (e) $\Upsilon_\ell = \text{tcf}(\prod_{\varepsilon < \kappa} \mu_\varepsilon^{+\ell})$ for $\ell \in \{0, 1, 2\}$
- (f) $\mu^+, 2^\mu \notin \{\Upsilon_0, \Upsilon_1, \Upsilon_2\}$

Proof.

By the constructions of [18],[19], and mainly [20].

□_{4.2}

Applying the above theorem to our problems, yields the following:

Corollary 4.3. *Strong polarized relations for singular cardinals.*

Suppose μ is a supercompact cardinal, GCH holds above μ and $\delta < \mu_0 < \mu$. There is a forcing notion \mathbb{P} as in the theorem above, making $\mu = \mu_0^{+\delta}$, such that:

- (\aleph) The strong relation $(\mu^+)_\mu \rightarrow (\mu^+)_2^{1,1}$ holds in $\mathbf{V}^{\mathbb{P}}$.
- (\beth) The strong relation $(2^\mu)_\mu \rightarrow (2^\mu)_2^{1,1}$ holds in $\mathbf{V}^{\mathbb{P}}$.

□_{4.3}

Further corollaries, in this light, can be obtained (e.g., for getting an entirely positive spectrum of cardinals between μ^+ and 2^μ , with respect to the strong polarized relation). We also indicate to an interesting consistency with GCH. By generalizing a remark of Foreman (quoted in [22]), we can deduce that the consistency of the relation $(\mu^+)_\mu \rightarrow (\mu^+)_2^{1,1}$ under the negation of GCH implies the consistency of $(\mu^+)_\mu \rightarrow (\alpha)_2^{1,1}$ for every $\alpha < \mu^+$ with GCH. So this consistency holds, in light of the theorems above, for each singular cardinal.

REFERENCES

1. James E. Baumgartner and Richard Laver, *Iterated perfect-set forcing*, Ann. Math. Logic **17** (1979), no. 3, 271–288. MR 556894 (81a:03050)
2. Andreas Blass, *Combinatorial cardinal characteristics of the continuum*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 395–489. MR 2768685
3. Andreas Blass and Saharon Shelah, *There may be simple P_{\aleph_1} - and P_{\aleph_2} -points and the Rudin-Keisler ordering may be downward directed*, Ann. Pure Appl. Logic **33** (1987), no. 3, 213–243. MR 879489 (88e:03073)
4. ———, *Ultrafilters with small generating sets*, Israel J. Math. **65** (1989), no. 3, 259–271. MR 1005010 (90e:03057)
5. P. Erdős, A. Hajnal, and R. Rado, *Partition relations for cardinal numbers*, Acta Math. Acad. Sci. Hungar. **16** (1965), 93–196. MR MR0202613 (34 #2475)
6. P. Erdős and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489. MR MR0081864 (18,458a)
7. Sakae Fuchino, Noam Greenberg, and Saharon Shelah, *Models of real-valued measurability*, Ann. Pure Appl. Logic **142** (2006), no. 1-3, 380–397. MR 2250550 (2007d:03091)
8. Shimon Garti and Saharon Shelah, *Strong polarized relations for the continuum*, Annals of Combinatorics **16** (2012), 271–276.
9. ———, *Combinatorial aspects of the splitting number*, Annals of Combinatorics (accepted).
10. ———, *A strong polarized relation*, The Journal of Symbolic Logic (accepted).
11. ———, *Partition calculus and cardinal invariants*, The Journal of the Mathematical Society of Japan (submitted).
12. Andras Hajnal and Jean L. Larson, *Partition relations*, Handbook of set theory, Springer, Dordrecht, 2010, pp. 129–213.
13. Albin L. Jones, *On a result of Szemerédi*, J. Symbolic Logic **73** (2008), no. 3, 953–956. MR 2444279 (2009i:03045)
14. ———, *A polarized partition relation for cardinals of countable cofinality*, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1445–1449. MR 2367118 (2008k:03091)
15. R. Laver, *Partition relations for uncountable cardinals $\leq 2^{\aleph_0}$* , Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, North-Holland, Amsterdam, 1975, pp. 1029–1042. Colloq. Math. Soc. Janós Bolyai, Vol. 10. MR 0371652 (51 #7870)
16. Richard Laver, *Making the supercompactness of κ indestructible under κ -directed closed forcing*, Israel J. Math. **29** (1978), no. 4, 385–388. MR 0472529 (57 #12226)
17. Azriel Lévy, *Definability in axiomatic set theory. II*, Mathematical Logic and Foundations of Set Theory (Proc. Internat. Colloq., Jerusalem, 1968), North-Holland, Amsterdam, 1970, pp. 129–145. MR 0268037 (42 #2936)
18. Menachem Magidor, *On the singular cardinals problem. I*, Israel J. Math. **28** (1977), no. 1-2, 1–31. MR 0491183 (58 #10449a)
19. ———, *On the singular cardinals problem. II*, Ann. of Math. (2) **106** (1977), no. 3, 517–547. MR 0491184 (58 #10449b)
20. Saharon Shelah, *The singular cardinals problem: independence results*, Surveys in set theory, London Math. Soc. Lecture Note Ser., vol. 87, Cambridge Univ. Press, Cambridge, 1983, pp. 116–134. MR 823777 (87b:03114)
21. ———, $\text{CON}(u > i)$, Arch. Math. Logic **31** (1992), no. 6, 433–443. MR 1175937 (93i:03066)
22. ———, *A polarized partition relation and failure of GCH at singular strong limit*, Fund. Math. **155** (1998), no. 2, 153–160. MR MR1606515 (99b:03064)
23. ———, *Proper and improper forcing*, second ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998. MR 1623206 (98m:03002)

24. Robert M. Solovay, *Real-valued measurable cardinals*, Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967), Amer. Math. Soc., Providence, R.I., 1971, pp. 397–428. MR 0290961 (45 #55)
25. Jindřich Zapletal, *Splitting number at uncountable cardinals*, J. Symbolic Logic **62** (1997), no. 1, 35–42. MR 1450512 (98c:03099)

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